Elementary Mathematics:
Properties of Number Ranges

Proof of the Irrationality of the Square Root of 2 ($\sqrt{2} \in \mathbb{R}$)

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There are various reasonings for the proof of the irrationality of the square root of 2. This article explains exhaustively the indirect proof variant and shows one of the possible abbreviated forms in mathematical notation. Exercises with solutions consolidate the acquired knowledge.

Contents

1 The Indirect Proof

1.1 The Approach with the Indirect Proof

The "law of the excluded middle" (Tertium non datur) states, that a statement and its negation cannot both be false and that one of both must be true.

In the indirect reasoning it is assumed, that the assertion to be proven is false. Thus, we assume, that the negation of the statement is true. Then, this assumption is lead to a contradiction.

However, if the negation of a statement leads to a contradiction, then the statement must be true. According to the *law of the excluded middle*, there can be no third possibility.

Besides the binary logic, there are also polyvalent resp. n-valued logics.[1]

1.2 On the Notation of Proofs

Proof problems often begin with "Show that" and "Proove that" resp. "Proove:". In German texts you also find "Z. z.:", the abbreviated form of "Zu zeigen:" ("To show:").

In indirect reasonings the emerging contradicton is denoted with a lightning flash: ↯.

The closure of a proof is either marked with the abbreviation "q.e.d", which stands for the latin words *quod erat demonstrandum*, in English: "which is what had to be proven". Or a white resp. black square is used as closure symbol: □.

1.3 Definition W. l. o. g.

W. l. o. g., "Without loss of generality" is a hint used in mathematical proofs which says, that the special case considered within the proof also covers all other possible cases.

Thus, you calculate exemplarily one case which stands simultaneously for all other possiblities. The case treated within the proof therefore also is charactistic and valid for all other cases, without exception.

2 Proof of the Irrationality of the Square Root of Two

2.1 Introduction

We want to show, that $\sqrt{2}$ is not describable as fraction, that $\sqrt{2} \notin \mathbb{Q}$.

[1] Foundational work: Gotthard Günther, "Idee und Grundriß einer nicht-Aristotelischen Logik. 1. Band: Die Idee und ihre philosophischen Voraussetzungen.", Verlag von Felix Meiner, Hamburg 1959.

Fractions have the form $\frac{a}{b}$ and come from the set of the rational numbers, $\mathbb{Q}$. In the numerator is a whole number.[2] The denominator b must be a natural number,[3] so that a division by zero is excluded:

$$\mathbb{Q} := \left\{ \frac{a}{b} \,\middle|\, a \in \mathbb{Z} \text{ and } b \in \mathbb{N} \right\}$$

Read: "Q is the set of all elements (all fractions) a over b such that a is the element of Z and b is the element of N".

We now show, that $\sqrt{2}$ does not belong to the set of the rational numbers, but to the superset of $\mathbb{Q}$, the set of the real numbers, $\mathbb{R}$.

2.2 Proof of the Irrationality of $\sqrt{2}$

Show: $\sqrt{2}$ is irrational.

Proof: indirect. We assume, that $\sqrt{2}$ is *not* irrational.

1. Then $\sqrt{2}$ is rational ($\in \mathbb{Q}$) and can be written as fraction:

$$\sqrt{2} = \frac{a}{b}$$

2. Without loss of generality[4] we assume, that $\frac{a}{b}$ is presented as reduced fraction, since

[2] The set of integers: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

[3] The set of natural numbers without zero: $\mathbb{N} = \{1, 2, 3, \ldots\}$.

[4] A general definition on W. l. o. g. can be found in the section 1.3 on the facing page. Explanation in the context of our case: The assumption, that the fraction $\frac{a}{b}$ can be reduced to prime factors holds true for all other possible cases, that is all other roots of *nonquadratic numbers*, such as $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, etc. Read the following explanation on why only roots of non quadratic numbers are permitted, not before the internalization of the proof within the main text.

Would we also permit quadratic numbers, then the assumption, that the square root of the quadratic number can be described as fraction of two coprime numbers would *not* lead to a contradiction.

Example: The 4 is a quadratic number, $4 = 2 \cdot 2$, and can be described as fraction, for instance $\sqrt{4} = \frac{8}{4} = \frac{2}{1} = 2$.

$$\begin{aligned} \sqrt{4} &= \tfrac{a}{b} && \mid (\ )^2 \text{ squaring} \\ 4 &= \tfrac{a^2}{b^2} && \mid \cdot b^2 \\ 4b^2 &= a^2 \end{aligned}$$

From the equation $4b^2 = a^2$ arises, that a^2 must be an even number, thus the numerator is even. Therefore we equate $a = 2k$, solve for b^2 and ascertain, that b is an uneven number and therefore *no* contradiction emerges:

$$\begin{aligned} 4b^2 &= (2k)^2 \\ 4b^2 &= 4k^2 && \mid \div 4 \\ b^2 &= k^2 && \mid \sqrt{\ } \\ b &= k \end{aligned}$$

to each fraction there also exists a reduced fraction. That means, that a and b are relatively prime, that they have no other common denominator than the 1.[5]

3. We now transform the equation to a^2:

$$\begin{array}{rcll} \sqrt{2} &=& \frac{a}{b} & \mid (\ )^2 \text{ squaring} \\ 2 &=& \left(\frac{a}{b}\right)^2 & \\ 2 &=& \frac{a^2}{b^2} & \mid \cdot b^2 \\ 2b^2 &=& a^2 & \end{array}$$

It is clear from the equation $2b^2 = a^2$, that a^2 must be an even number. An even number namely can be defined as $\pm 2k$, with $k \in \mathbb{N}_0$.[6]

So the 2 is a divisor of a^2, is contained in the set of divisors of a^2.[7]

4. We now use the following elementary facts for our further reasoning: If a number is even, then its square is also even.[8] If a number is odd, then its square is also odd.[9]

From a^2 being even therefore follows, that a must be also an even number.[10]

5. As the 2 is a divisor of a, thus an even number, we can replace the a in $2b^2 = a^2$ with $2k$ ($k \in \mathbb{N}_0$):

[5] Examples for (reduced) coprime fractions: $\frac{6}{10} = \frac{3}{5}$, the greatest common divisor of 3 and 5 is the 1: gcd(3,5) = 1. $\frac{30}{5} = \frac{6}{1}$, gcd(6,1) = 1. For $\frac{17}{13}$ holds: gcd(17,13) = 1. The cases with an 0 in the numerator $\{\frac{0}{1}, \frac{0}{2}, \frac{0}{3}, \ldots\}$ are hereby considered, the zero is divisible by all numbers: $\forall x \in \mathbb{N}$ (in general $\forall x \in \mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$) holds true "x divides 0". Fractions of the form $\frac{0}{x}$ (with $x \in \mathbb{N}$) no longer can be reduced. As in the numerator holds $1 \mid 0$ ("1 divides 0") and in the numerator $1 \mid x$, $\frac{0}{x}$ is always coprime, has only the 1 as greatest common divisor: $\gcd(0, x) = 1$.

[6] Even numbers are multiples of two, they have the form $2 \cdot k$. The set of the possible natural numbers for k also comprises the zero ($k \in \mathbb{N}_0$), so that, besides positive and negative even numbers, the case $2 \cdot 0 = 0$ is also possible. The zero is also considered as an even number. For our b^2 naturally nevertheless holds, that $b^2 \geqq 1$, since $b > 0$ must apply ($b \in \mathbb{N}$), for that no division by zero can happen.

[7] $2 \mid a^2$ resp. $2 \in \mathrm{T}_{a^2}$.

[8] Examples for $x = (2k)^2$, with $k \in \mathbb{N}$: $x = 2, x^2 = 4$; $x = 4, x^2 = 16$; $x = 6, x^2 = 36$.

[9] Examples for $x = (2k+1)^2$, with $k \in \mathbb{N}_0$: $x = 1, x^2 = 1$; $x = 3, x^2 = 9$; $x = 5, x^2 = 25$.

[10] In *even* numbers, each prime factor ($p \geqq 2$) of a must be contained at least twice in a^2, and even numbers also always contain the 2 as prime factor. Examples:
$a = 2\,(= 2), a^2 = 4\,(= 2 \cdot 2)$;
$a = 4\,(= 2 \cdot 2), a^2 = 16\,(= 2 \cdot 2 \cdot 2 \cdot 2)$;
$a = 6\,(= 2 \cdot 3), a^2 = 36\,(= 2 \cdot 2 \cdot 3 \cdot 3)$;
$a = 8\,(= 2 \cdot 2 \cdot 2), a^2 = 64\,(= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2)$;
$a = 10\,(= 2 \cdot 5), a^2 = 100\,(= 2 \cdot 2 \cdot 5 \cdot 5)$
$a = 30\,(= 2 \cdot 3 \cdot 5), a^2 = 900\,(= 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 5)$.

$$\begin{aligned} 2b^2 &= a^2 \\ 2b^2 &= (2k)^2 \\ 2b^2 &= 4k^2 \quad \mid :2 \\ b^2 &= 2k^2 \ \lightning \end{aligned}$$

In the last line the contradiction has now emerged: If $b^2 = 2k^2$, then b^2 and thus also b are even. However, in our assumption, we started from the premise that our fraction $\frac{a}{b}$ is already coprime, meaning that it is there in reduced form. If, however, the numerator *and* the denominator are both even, then you still can cancel the fraction.[11]

Thus the assumption, that $\sqrt{2}$ is not irrational, is false. According to the *law of the excluded middle* we thereby have proven, that the first statement is true: $\sqrt{2}$ is irrational.

□

2.3 Proof in abbreviated Form

When authoring mathematical texts it is beneficial to note the contents in parallel also in pure mathematical notation. Thereby the document becomes accessible to an international audience.

Set notations can vary, moreover, irrational numbers are sometimes summarized in an own set, denoted with a capital $\mathbb{I}$: $\mathbb{I} = \{\sqrt{2}, \pi, \dots\}$.[12]

2.3.1 Explanation of the Mathematical Symbols

The sign for divisibility, $\mid$. Example: $2 \mid 4$, read "2 divides 4". $2 \nmid 5$, read "2 does not divide 5". Generally: $a \mid b$, "a divides b". $a \nmid b$, "a does not divide b".

Coprimeness, $\perp$. Example: $3 \perp 5$, read "3 is relatively prime to 5" resp. "3 is coprime to 5" resp.. "3 and 5 are mutually prime". $3 \not\perp 6$, read "3 is not relatively prime to 6" resp. "3 is not coprime to 6" resp. "3 and 6 are not mutually prime". Generally: $a \perp b$, "a is relatively prime to b" resp. "a is coprime to b" resp. "a and b are mutually prime". $a \not\perp b$, read "a is not relatively prime to b" resp. "a is not coprime to b" resp. "a and b are not mutually prime".[13]

The universal all quantifier $\forall$ originates from the propositional calculus and is read as "for all". Example: $\forall x \in A$: $2 \mid x$.

The implication, $\Rightarrow$. Example: $A \Rightarrow B$, read "if A then B" resp. "A implies B".

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[11] Examples for unreduced even fractions $\frac{a}{b}$, with $a \in \mathbb{Z}$ und $b \in \mathbb{N}$: $\{\dots, \frac{-4}{4}, \frac{-2}{4}, \dots, \frac{-4}{2}, \frac{-2}{2}, \frac{2}{2}, \frac{4}{2}, \dots, \frac{2}{4}, \frac{4}{4}, \dots\}$.

[12] In this article, mainly the sanserif variant of "dsfont" is used for sets of numbers, following the double bar notation when writing on blackboards: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$. The font (the character set) "amsb" is widespread: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$. In older literature, mostly Fraktur letters are used as symbols for sets of numbers.

[13] In geometry, the symbol $\perp$ indicates orthogonality (perpendicularity) between vectors. Example: $\vec{a} \perp \vec{b}$, "$\vec{a}$ is orthogonal to $\vec{b}$".

The logical equivalence, $\Leftrightarrow$. Example: $A \Leftrightarrow B$, read "A is equivalent to B" resp. "A and B are equivalent."

W. l. o. g., "without loss of generality": See section 1.3 on page 2 and footnote 4.

Partially, there are additional ways to read the expressions.

2.3.2 Proof $\sqrt{2} \in \mathbb{R}$, in abbreviated form

Next up, one of several possible transcripts of the proof in abbreviated form.

Show $\sqrt{2}$ is irrational ($\sqrt{2} \in \mathbb{R}$).

Proof: indirect. We assume, that $\sqrt{2}$ is not irrational:

1. Assumption: $\sqrt{2}$ is not irrational $\Rightarrow \sqrt{2} \in \mathbb{Q}$, i.e. $\sqrt{2} = \frac{a}{b}$.
2. Wlog: $a \perp b$.
3. $\sqrt{2} = \frac{a}{b} \Leftrightarrow 2b^2 = a^2$.
4. $(2b^2 = a^2) \Rightarrow 2 \mid a^2$, as $a^2 = \pm 2k$, with $k \in \mathbb{N}_0$.
5. $\forall x$ with $x = 2k$, $k \in \mathbb{N} : x^n = 2k$ rsp. $2 \mid x^n$.
6. $\Rightarrow 2 \mid a$, and: $2b^2 = a^2 \Leftrightarrow 2b^2 = (2k)^2$.
7. $\Rightarrow b^2 = 2k^2 \Rightarrow a \not\perp b \Rightarrow \lightning$ to 2.

□

3 Exercises with Solutions

Of the following exercises, the first four are taken from [1].[14]

1. Show: $\sqrt{3}$ is irrational.

Proof: indirect.

1. Assumption: $\sqrt{3} \in \mathbb{Q}$. Then follows: $\sqrt{3} = \frac{a}{b}$ with $a \in \mathbb{Q}$, $b \in \mathbb{N}$; a and b are reduced and coprime.

2.

$$\begin{aligned} \sqrt{3} &= \frac{a}{b} && \mid (\ )^2 \text{ squaring} \\ 3 &= \frac{a^2}{b^2} && \mid \cdot b^2 \\ 3b^2 &= a^2 && (1) \end{aligned}$$

3. Isolating the numerator variable: From $3b^2 = a^2$ follows $3 \mid a^2$, as the 3 in $3 \cdot b^2 = a^2$ is divisor (prime factor) of a.

[14] Page 99, exercises 212, a) till d)

4. From $3 \mid a^2$ follows $3 \mid a$:

$$\begin{aligned} 3b^2 &= a^2 \\ 3b^2 &= a \cdot a \quad \mid \div a \\ \tfrac{3b^2}{a} &= a \\ 3 \cdot \left(\tfrac{b^2}{a}\right)^2 &= a \end{aligned}$$

$$\Rightarrow 3 \mid a$$

We denote the coefficient of the 3 with k

$$3 \cdot \underbrace{\left(\frac{b^2}{a}\right)}_{:= k} = a$$

Thus, $3k = a$.

5. Isolating the denominator variable: We now insert our result for a in (1).

$$\begin{aligned} 3b^2 &= a^2 \\ 3b^2 &= (3k)^2 \\ 3b^2 &= 9k^2 \quad \mid \div 3 \\ b^2 &= 3k^2 \end{aligned}$$

$\Rightarrow 3 \mid b^2$, as the 3 is a prime factor of b^2:

$$b^2 = \underbrace{3}_{\text{divisor}} \cdot k$$

6. From $3 \mid b^2 \Rightarrow 3 \mid b$:

$$\begin{aligned} b^2 &= 3k^2 \quad \mid \div b \\ b &= 3 \cdot \frac{k^2}{b} \end{aligned}$$

$$\Rightarrow 3 \mid b \quad ↯$$

$3 \mid a$ and $3 \mid b$ is a contradiction to our assumption, that a and b are relatively prime.

$\Rightarrow \sqrt{3} \notin \mathbb{Q}$.

$\square$

2. Proove: $\sqrt{6} \notin \mathbb{Q}$.

Proof: indirect.

1. Assumption: $\sqrt{6} \in \mathbb{Q}$.

2. Then follows ($\sqrt{6} = \sqrt{2 \cdot 3} = \sqrt{2} \cdot \sqrt{3}$), a and b are reduced and coprime:

$$\begin{aligned} \sqrt{2} \cdot \sqrt{3} &= \frac{a}{b} \\ \sqrt{2} \cdot \sqrt{3} &= \frac{a}{b} \quad \mid (\ )^2 \\ (\sqrt{2} \cdot \sqrt{3})^2 &= \left(\frac{a}{b}\right)^2 \\ \sqrt{2} \cdot \sqrt{2} \cdot \sqrt{3} \cdot \sqrt{3} &= \frac{a^2}{b^2} \\ 2 \cdot 3 &= \frac{a^2}{b^2} \quad \mid \cdot b^2 \\ 2 \cdot 3 \cdot b^2 &= a^2 \end{aligned}$$

The 2 is a divisor of a^2: $2 \mid a^2$.

3. Isolating the numerator variable:

$$\begin{aligned} 2 \cdot 3 \cdot b^2 &= a^2 \quad \mid \div a \\ \frac{2 \cdot 3 \cdot b^2}{a} &= a \\ \frac{2}{1} \cdot \frac{3 \cdot b^2}{a} &= a \\ \underbrace{2}_{\text{divisor of } a} \cdot \frac{3b^2}{a} &= a \end{aligned}$$

$\Rightarrow 2 \mid a$, the numerator is even.

4. Isolating the denominator variable: a is also an even number, i.e. $a = 2k$.

$$\begin{aligned} 6b^2 &= a^2 \\ 6b^2 &= (2k)^2 \\ 6b^2 &= 4k^2 \quad \mid \div 2 \\ 3b^2 &= 2k^2 \end{aligned}$$

From $3 \underbrace{\cdot \ b^2}_{\text{even part}} = 2k^2$ follows $2 \mid b^2$.

5.

$$\begin{aligned} 3b^2 &= 2k^2 \quad \mid \div b \\ \frac{3b^{\not 2}}{\not b} &= \frac{2k^2}{b} \quad \mid \div 3 \\ b &= \frac{2k^2}{b \cdot 3} \\ b &= \frac{2}{1} \cdot \frac{k^2}{3b} \\ b &= 2 \cdot \frac{k^2}{3b} \\ &\Rightarrow 2 \mid b \ \lightning \end{aligned}$$

If $2 \mid a$ and $2 \mid b$ are true, then the numerator and the denominator have a commmon divisor. That contradicts the assumption, that a and b are coprime.

q. e. d.

3. Show: $\sqrt[3]{5} \notin \mathbb{Q}$.

Proof: indirect.

1. Assumption: $\sqrt[3]{5}$ is rational, a and b are reduced and relatively prime. We rearrange the equation and isolate subsequently the numerator variable and the denominator variable.

$$\sqrt[3]{5} = \frac{a}{b} \quad | \ (\ )^3$$
$$5 = \frac{a^3}{b^3} \quad | \cdot b^3$$
$$5b^3 = a^3 \qquad (1)$$

2. Isolating the numerator variable:

$$5 \cdot b^3 = a^3$$

$$\Rightarrow 5 \text{ is prime divisor of } a^3 : 5 \mid a^3$$

$$5 \cdot b^3 = a^3 \quad | \div a^2$$
$$5 \cdot \frac{b^3}{a^2} = a$$
$$\Rightarrow 5 \mid a$$

3. Isolating the denominator variable: From (1) we know, that 5 is a divisor of $a^3 (5 \mid a^3)$, therefore we equate $a^3 = 5k$.

$$5b^3 = a^3$$
$$5b^3 = (5k)^3$$
$$5b^3 = 5^3 \cdot k^3$$
$$5b^3 = 125k^3 \quad | \div 5$$
$$b^3 = 25 \cdot k^3$$
$$b^3 = 5 \cdot 5 \cdot k^3$$

$$\Rightarrow 5 \mid b^3$$

4.

$$b^3 = 25k^3 \quad | \div b^2$$
$$\frac{b^{\cancel{3}1}}{\cancel{b^2}} = 25 \cdot \frac{k^3}{b^2}$$
$$b = 5 \cdot 5 \cdot \frac{k^3}{b^2}$$

$$\Rightarrow 5 \mid b \ ↯$$

As $5 \mid a$ and $5 \mid b$ is true, a and b are not relatively prime.

■

4. Show: $\sqrt[3]{6}$ is irrational.

Proof: indirect.

Assumption: $\sqrt[3]{6}$ is rational. Then one has,with a and b reduced and relatively prime: $\sqrt[3]{6} = \frac{a}{b}$, $a \in \mathbb{Z}$ und $b \in \mathbb{N}$.

1. Rearranging and isloating of the numerator variable:

$$\sqrt[3]{6} = \frac{a}{b}$$

$$\sqrt[3]{6} = \frac{a}{b} \quad | \ (\ )^3$$
$$6 = \left(\frac{a}{b}\right)^3$$
$$6 = \frac{a^3}{b^3} \quad | \cdot b^3$$
$$6b^3 = a^3 \qquad (1)$$
$$(2 \cdot 3) \cdot b^3 = a^3 \quad |\ ^3$$

$$\frac{a^{\cancel{3}}}{\cancel{a^2}} = (2 \cdot 3) \cdot \frac{b^3}{a^2}$$

$$a = 3 \cdot \underbrace{\left(2 \cdot \frac{b^3}{a^2}\right)}_{:=k} \qquad (2)$$

$\Rightarrow 3 \mid a$, as the 3 is a prime divisor of a.

2. Isolating the numerator variable b: From (2) follows $a = 3k$. Inserting into (1):

$$6b^3 = a^3$$
$$6b^3 = (3k)^3$$
$$6b^3 = 27k^3 \quad \mid \div 3$$
$$2b^3 = 9k^3 \quad \mid \div 2$$

$$b^3 = \frac{9}{2} \cdot k^3 \quad \mid \div b^2$$
$$b = \frac{9}{2} \cdot \frac{k^3}{b^2}$$
$$b = \frac{3}{1} \cdot \frac{3}{1} \cdot \frac{1}{2} \cdot \frac{k^3}{b^2}$$
$$b = 3 \cdot \left(\frac{3}{2} \cdot \frac{k^3}{b^2}\right)$$

$\Rightarrow 3 \mid b$ ↯

As numerator and denominator, a and b, are not, as assumed, relatively prime and the $\sqrt[3]{6}$ cannot be described as a fraction, the $\sqrt[3]{6}$ must be, according to the law of the excluded middle, irrational: $\sqrt[3]{6} \in \mathbb{I}$.

$\sqrt[3]{6} \notin \mathbb{Q}$ bzw. $\sqrt[3]{6} \in \mathbb{I}$ resp. $\sqrt[3]{6} \in \mathbb{R}$.

□

5. Show: $\sqrt{7} \in \mathbb{I}$.

Proof: indirect.
1. Assumption: $\sqrt{7} \in \mathbb{Q}. \Rightarrow \sqrt{7} = \frac{a}{b}$.
2. W.l.o.g.: $a \perp b$.
3. $\sqrt{7} = \frac{a}{b} \overset{()^2}{\Leftrightarrow} 7 = \frac{a^2}{b^2} \overset{\cdot b^2}{\Leftrightarrow} 7b^2 = a^2 \overset{\div a}{\Leftrightarrow} 7 \cdot \frac{b^2}{a} = a \overset{k:=\frac{b^2}{a}}{\Leftrightarrow} 7 \cdot k = a \Rightarrow 7 \mid a$.
4. $7 \cdot b^2 = a^2 \overset{a:=7\cdot k}{\Longleftrightarrow} 7b^2 = (7k)^2 \Leftrightarrow 7b^2 = 49k^2 \overset{\div 7}{\Leftrightarrow} b^2 = 7k^2 \overset{\div b}{\Leftrightarrow} b = 7 \cdot \frac{k^2}{b} \Rightarrow 7 \mid b$.
5. $\Rightarrow a \not\perp b \Rightarrow$ ↯ to 2.

□

6. Show: $\sqrt{11} \notin \mathbb{Q}$.

Proof: indirect.

1. Assumption: $\sqrt{11} \in \mathbb{Q}. \Rightarrow \sqrt{11} = \frac{a}{b}$.
2. W.l.o.g.: $a \perp b$.
3. $\sqrt{11} = \frac{a}{b} \overset{()^2}{\Leftrightarrow} 11 = \frac{a^2}{b^2} \overset{\cdot b^2}{\Leftrightarrow} 11b^2 = a^2 \overset{\div a}{\Leftrightarrow} 11 \cdot \frac{b^2}{a} = a \overset{k:=\frac{b^2}{a}}{\Leftrightarrow} 11 \cdot k = a \Rightarrow 11 \mid a$.

4. $11 \cdot b^2 = a^2 \overset{a:=11\cdot k}{\Longleftrightarrow} 11b^2 = (11k)^2 \Leftrightarrow 11b^2 = 121k^2 \overset{\div 11}{\Leftrightarrow} b^2 = 11k^2 \Leftrightarrow b = 11 \cdot \frac{k^2}{b}$
   $\Rightarrow 11 \mid b$.
5. $\Rightarrow a \not\perp b \Rightarrow \lightning$ zu 2.

$\square$

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3.4 Classic Typset in highest Quality, with Free Software

TeX [tεχ][15] and the macrosystem LaTeX[16] form the most widespread typset system in mathematics, in physics and computer science as well as in numerous further diciplines.

"The Beauty of LATEX"[17] informs about the advantages. PDF example documents can be find i.a. at the TeX User Group[18] and in a compilaton of the Association of American University Presses, presented on tsengbooks.com[19]

For the majority of the computer users it could be unusual to create documents with a text editor and with macro commands. With the cross-platform available graphic interface LyX[20], you can use LaTeX as easy as a word processor.

Word processors, (e. g. LibreOffice)[21] and graphical DTP software, (e. g. Scribus)[22], mainly work by following the WYSIWYG-principle.[23] By contrast, LaTeX and LyX base on a markup language.[24] The formatting of text areas, headings and other outlinings hereby is done indirectly (WYSIWYM).[25]. Meanwhile, word processors partly offer similar functionalities, via so called style templates.

The installment of LyX should precede the installation of a *full* TeX distribution, thereby the download of packages becomes unnecessary.

Become acquainted with LyX. You may want to try as document class, e. g., KOMAScript, "(KOMA-Script)"and "book (KOMA-Script)", together with "Latin Modern fonts" as standard font. Find your prefered fonts/character sets in the LaTeX Font Catalogue.[26] Thousands of packages offer comprehensive possibilities for numerous professions and application areas; also own templates can be written, all typset details are possible.

Calligraphy fonts, Fraktur, Sütterlin and numerous additional fonts and styles can be used universally, e.g. as elaborate shaped titles for websites, book covers, film and music albums, posters, greeting cards and gift cards. The openness of the TeX- and LaTeX-base enabled/gained a momentum resp.

[15] en.wikipedia.org/wiki/TeX
[16] latex-project.org/intro.html
[17] nitens.org/taraborelli/latex
[18] tug.org/texshowcase
[19] tsengbooks.com
[20] lyx.org
[21] libreoffice.org
[22] scribus.net
[23] en.wikipedia.org/wiki/WYSIWYG
[24] Markup language
[25] en.wikipedia.org/wiki/WYSIWYM
[26] tug.dk/FontCatalogue

an Eigendynamik, beauty, functionality and expressiveness, to which, worldwide, countless nations, profession groups and private persons contribute permanently. Donald Knuth's[27] TeX interpreter belongs to the mightiest computer typeset systems and, moreover, is the most archive safe and most durable, that ever has been created, as far as our civilization knows. His cultural meaning is rightly equated with the invention of the letterpress printing by Gutenberg – free Software, emphasizing the beauty and freedom of individuals and nations, supporting descriptive linguistic science[28].

This document has been created under Ubuntu GNU/Linux 16.04 LTS, with TeX, LaTeX, KOMA-Script, the font "Latin Modern" and the editor Kate.

Learn more about free typsetting software (in German) under "Schriftsatzprogramme für Druckvorstufe und Internet".[29]

[27] www-cs-faculty.stanford.edu/~uno

[28] www.schriftdeutsch.de/orth-li1.htm

[29] www.peterjockisch.de/Empfehlungen-zu-Freier-Software/Empfehlungen-zu-Freier-Software.html#Verweisliste-Freie-Schriftsatzprogramme